interpretation of $\exists i{\in}I$ is changed. In fact, the interpretation of $\exists i{\in}I\ M_i$ (with each M_i a singleton) now becomes $[\coprod_{i\in I} M_i]$, where for each set X, $[X] = \{u\colon u = 0 \wedge \exists x.\ x \in X\}$ is the *canonical singleton* associated with X.

It follows that, under the "formulas-as-monotypes" interpretation, the proposition $\forall i{\in}I\ \exists j{\in}J\ M_{ij}$ is interpreted as the singleton

(1) $$\prod_{i\in I}[\coprod_{j\in J} M_{ij}]$$

and the proposition $\exists f{\in}J^I\ \forall i{\in}I\ M_{if(i)}$ as the singleton

(2) $$[\coprod_{f\in J^I}\prod_{i\in I} M_{if(i)}].$$

Under the "formulas-as-monotypes" interpretation AC would be construed as asserting the existence of an isomorphism between (1) and (2).

Now it is readily seen that to give an element of (1) amounts to no more than affirming that, for every $i \in I$, $\bigcup_{j\in J} M_{ij}$ is nonempty. But to give an element of (2) amounts to specifying maps $f \in J^I$ and g with domain I such that $\forall i{\in}I\ g(i) \in M_{if(i)}$. It follows that to assert the existence of an isomorphism between (1) and (2), that is, to assert AC under the "formulas-as-monotypes" interpretation, is tantamount to asserting AC in its usual form, so leading in turn to classical logic. This is in sharp contrast with AC under the "propositions-as-types" interpretation, where its assertion is automatically correct and so has no nonconstructive consequences.

equivalent. If we think of (the objects of) **Set** as representing simple or static types, then (the objects of) **Indset**, and hence also of $\mathbf{Set}^{\rightarrow}$, represent dependent or variable types. It is easily seen that a monotype, or object, in **Indset**, is precisely an object M for which each M_i has at most one element. Moreover, under the equivalence between **Indset** and $\mathbf{Set}^{\rightarrow}$, such an object corresponds to a monic map- object in $\mathbf{Set}^{\rightarrow}$.

Now consider $\mathbf{Set}^{\rightarrow}$ as a topos. Under the topos-theoretic interpretation in $\mathbf{Set}^{\rightarrow}$, formulas correspond to monic arrows, which in turn correspond to mono-objects in **Indset.** Carrying this over entirely to **Indset** yields the sought modification of the "propositions-as-types" paradigm to bring it into line with the topos-theoretic interpretation of formulas, namely, to take formulas or propositions to correspond to *mono*-objects, rather than to *arbitrary* objects. Let us call this the "formulas-as-monotypes" interpretation.

Finally let us reconsider AC under the "formulas-as-monotypes" interpretation within **Set.** In the "propositions-as-types" interpretation as applied to **Set**, the universal quantifier $\forall i \in I$ corresponds to the product $\prod_{i \in I}$ and the existential quantifier $\exists i \in I$ to the coproduct, or disjoint sum, $\coprod_{i \in I}$. Now in the "formulas-as-monotypes" interpretation, in which formulas correspond to singletons, $\forall i \in I$ continues to correspond to $\prod_{i \in I}$, since the product of singletons is still a singleton. But the

On the other hand, AC interpreted in the usual way, that is, using the rules of topos semantics, can, as we have already observed, be presented in the form of the distributive law

$$\prod_{i\in I}\bigcup_{j\in J}A_{ij} = \bigcup_{f\in J^I}\prod_{i\in I}A_{if(i)}$$

And this distributive law, by Diaconescu's theorem, implies the law of excluded middle for propositions.

So we see again that AC interpreted à la "propositions as types" is (constructively) canonically true, while construed topos-theoretically it is anything but, since so construed its affirmation yields classical logic. This prompts the question: what modification needs to be made to the "propositions-as-types" paradigm so as to yield the topos-theoretic interpretation of AC? An answer (due to M.E. Maietti) to this question can be furnished within the general framework of variable-type theories through the use of so-called *monotypes* (or mono-objects), that is, (variable) types containing at most one entity or having at most one proof. In the category **Set** of ordinary sets, mono objects are *singletons*, that is, sets containing at most one element.

Monotypes correspond to monic maps. This can be illustrated concretely by considering the categories **Indset** of *indexed* sets and $\mathbf{Set}^{\rightarrow}$ of *bivariant* sets. The objects of **Indset** are indexed sets of the form $M = \{<i, M_i>: i \in I\}$ and those of $\mathbf{Set}^{\rightarrow}$ maps $A \rightarrow B$ in **Set**, with appropriately defined arrows in each case. It can be shown that these two categories are

$$\gamma \to a = 1,$$

whence

$$a \neq 1 \to \neg\gamma,$$

so that

$$a = 0 \to \neg\gamma.$$

And the conjunction of this with (3) gives $\gamma \vee \neg\gamma$, as claimed.

Finally, I want to discuss the status of the axiom of choice in *dependent type theory*. In the "propositions as types" framework we may take the axiom of choice to be the assertion

$$\forall i \in I\, \exists j \in J\, M_{ij} \leftrightarrow \exists f \in J^I\, \forall i \in I\, M_{if(i)}$$

where $<M_{ij}: i \in I, j \in J>$ is any doubly indexed family of propositions (or sets). Now as we know it is generally held by constructivists that AC is a valid principle. In the "propositions-as-types" interpretation this amounts to affirming the existence of an isomorphism between the types $(\prod x \in A)(\Sigma y \in B)C(x,y)$ and $(\Sigma f \in A \to B)(\prod x \in A)C(x,fx)$, where $\prod$ is the dependent product type, Σ is the indexed sum type, and $A \to B$ is $(\prod x \in A)B$. Let us consider this (again) within (intuitionistic) set theory. Here the isomorphism is manifested as the natural bijection between $\prod_{i \in I} \coprod_{j \in J} M_{ij}$ and $\coprod_{f \in J^I} \prod_{i \in I} M_{if(i)}$.

of formulas $\alpha(x)$, $\beta(x)$ we introduce the "relativized" ε-term $\varepsilon_x\alpha/\beta$ and the "relativized" ε-axioms

$$(1)\ \exists x\, \beta(x) \to \beta(\varepsilon_x\alpha/\beta) \qquad (2)\ \exists x\, [\alpha(x) \wedge \beta(x)] \to \alpha(\varepsilon_x\alpha/\beta).$$

That is, $\varepsilon_x\alpha/\beta$ may be thought of as an individual that satisfies β if anything does, and which in addition satisfies α if anything satisfies both α and β. Notice that the usual ε–term $\varepsilon_x\alpha$ is then $\varepsilon_x\alpha/x = x$. In the classical ε-calculus $\varepsilon_x\alpha/\beta$ may be defined by taking

$$\varepsilon_x\alpha/\beta = \varepsilon_y[[y = \varepsilon_x(\alpha \wedge \beta) \wedge \exists x\, (\alpha \wedge \beta)] \vee [y = \varepsilon_x\beta \wedge \neg\exists x\, (\alpha \wedge \beta)]].$$

But the relativized ε-scheme is not derivable in the intuitionistic ε-calculus since it can be shown to imply **LEM**. To see this, given a formula γ define

$$\alpha(x) \equiv x = 1 \qquad \beta(x) \equiv x = 0 \vee \gamma.$$

Write a for $\varepsilon_x\alpha/\beta$. Then we certainly have $\exists x \beta(x)$, so (1) gives $\beta(a)$, i.e.

$$(3) \qquad a = 0 \vee \gamma$$

Also $\exists x\, (\alpha \wedge \beta) \leftrightarrow \gamma$, so (2) gives $\gamma \to \alpha(a)$, i.e.

In fact it is easy to see that, if $\mathscr{P}$ is taken to be intuitionistic predicate logic, then a number of first-order assertions undemonstrable within $\mathscr{P}$, for instance $\exists x(\exists x\alpha(x) \rightarrow \alpha(x))$, are provable within $\mathscr{P}_\varepsilon$. More interesting is the fact that certain purely *propositional* assertions undemonstrable within $\mathscr{P}$ are rendered provable within $\mathscr{P}_\varepsilon$. These include Dummett's scheme $A \rightarrow B \vee B \rightarrow A$ and (hence) the intuitionistically invalid De Morgan law $\neg(A \wedge B) \rightarrow \neg A \vee \neg B.$ But, curiously, the Law of Excluded Middle does *not* become demonstrable as a result of passing from intuitionistic $\mathscr{P}$ to $\mathscr{P}_\varepsilon$.

This is related to the fact (remarked on above) that in deriving **LEM** from **AC** one requires the principle of Extensionality of Functions. The analogous principle within the ε-calculus is the *Principle of Extensionality for ε-terms*:

(Ext) $$\forall x[\alpha(x) \leftrightarrow \beta(x)] \rightarrow \varepsilon_\alpha = \varepsilon_\beta.$$

An argument similar to the derivation of **LEM** from **AC** given above yields **LEM** from (Ext) within the intuitionistic ε-calculus.

It is interesting to note that the use of (Ext) can be avoided in deriving **LEM** in the intuitionistic ε-calculus if one employs *relative* ε-*terms,* that is, allows ε to act on *pairs* of formulas, each with a *single* free variable. Here, for each pair the form of a postulate he called the *logical* ε*-axiom* or the *transfinite axiom.* To formulate this postulate he introduced, for each formula $\alpha(x)$, a term (an *epsilon term*) $\varepsilon_x\alpha$ or simply ε_α which, intuitively, is intended to name an indeterminate object satisfying $\alpha(x)$. The ε-axiom then takes the form

$$(\varepsilon) \qquad \exists x\alpha(x) \rightarrow \alpha(\varepsilon_\alpha).$$

All that is known about ε_α is that, if anything satisfies α, it does[2]. Now since α may contain free variables other than x, the identity of ε_α depends, in general, on the values assigned to these variables. So ε_α may be regarded as the result of having chosen, for each assignment of values to these other variables, a value of x so that $\alpha(x)$ is satisfied. That is, ε_α may be construed as a choice function, and the ε-axiom accordingly seen as a version of **AC.**

An ε-*calculus* $\mathscr{P}_\varepsilon$ is obtained by starting with a system $\mathscr{P}$ of first-order predicate logic, augmenting it with epsilon terms, and adjoining as an axiom scheme the formulas (ε). It is known that when $\mathscr{P}$ is classical predicate logic, $\mathscr{P}_\varepsilon$ is *conservative* over $\mathscr{P}$, that is, each assertion of $\mathscr{P}$ demonstrable in $\mathscr{P}_\varepsilon$ is also demonstrable in $\mathscr{P}$: the move from $\mathscr{P}$ to $\mathscr{P}_\varepsilon$ does not enlarge the body of demonstrable assertions in $\mathscr{P}$. But for *intuitionistic* predicate logic the situation is otherwise.

[2] David Devidi has the happy inspiration of calling ε_α "the thing most likely to be α."

$$A \vee \neg A,$$

that is, **LEM**.

Note that in deriving **LEM** from version **AC** essential use was made of the principles of Predicative Comprehension and Extensionality of Functions. It follows that, in systems of constructive mathematics affirming **AC** (but not **LEM**) *either the principle of Predicative Comprehension or the Principle of Extensionality of Functions must fail.* While the Principle of Predicative Comprehension can be given a constructive justification, no such justification can be provided for the principle of Extensionality of Functions. Functions on predicates are given intensionally, and satisfy just the corresponding Principle of Intensionality $\forall X\, \forall Y\, \forall F[X = Y \rightarrow FX = FY]$. The Principle of Extensionality can easily be made to fail by considering, for example, the predicates P: *rational featherless biped* and Q: *human being* and the function K on predicates which assigns to each predicate the number of words in its description. Then we can agree that $P \equiv Q$ but $KP = 3$ and $KQ = 2$.

2. *Hilbert's Epsilon Calculus.* In developing his theory of mathematical proof during the 1920s Hilbert had enlisted the Axiom of Choice as an essential tool in his defence of classical mathematics against the attacks of the intuitionists. In the logical calculus he developed the Axiom of Choice appears in

Let $\Phi(X)$ be the formula $X \equiv U \ \vee \ X \equiv V$. Then clearly we may assert $\forall X[\Phi(X) \rightarrow \exists x X(x)]$ so (***) may be invoked to assert $\exists F$ $\forall X[\Phi(X) \rightarrow X(FX)]$. Now we can introduce a function constant K together with the assertion

(2) $$\forall X[\Phi(X) \rightarrow X(KX)].$$

Evidently we may assert $\Phi(U)$ and $\Phi(V)$, so it follows from (2) that we may assert $U(KU)$ and $V(KV)$, whence also, using (1),

$$[A \vee KU = 0\,] \wedge\ [A \vee KV = 1].$$

Using the distributive law (which holds in intuitionistic logic), it follows that we may assert

$$A \vee [KU = 0\ \ \wedge KV = 1].$$

From the presupposition that $0 \neq 1$ it follows that

(3) $$A \vee KU \neq KV$$

is assertable. But it follows from (1) that we may assert $A \rightarrow U \equiv V$, and so also, using Extensionality of Functions, $A \rightarrow KU = KV$. This yields the assertability of $KU \neq KV \rightarrow \neg A$, which, together with (3) in turn yields the assertability of

(AC) $\forall X[\Phi(X) \rightarrow \exists x X(x)] \rightarrow \exists F\, \forall X[\Phi(X) \rightarrow X(FX)]$

may be taken as the axiom of choice in $\mathscr{L}$.

We assume that the background logic of $\mathscr{L}$is intuitionistic logic. Given certain mild further presuppositions, **AC** can be shown to imply **LEM**, the law of excluded middle that, for any for any proposition *A*, $A \vee \neg A$. These mild further presuppositions latter may be stated:

Predicative Comprehension $\exists X\, \forall x[X(x) \leftrightarrow \varphi(x)]$

Here φ is a formula not containing any bound predicate variables.

Extensionality of Functions $\forall X\, \forall Y\, \forall F[X \equiv Y \rightarrow FX = FY]$

Here $X \equiv Y$ is an abbreviation for $\forall x[X(x) \leftrightarrow Y(x)]$, that is, *X* and *Y* are *extensionally equivalent.*

In addition we assume the presence of two individuals 0 and 1. Their distinctness is expressed by means of the trivial presupposition $0 \neq 1$.

Now let *A* be a given proposition. By Predicative Comprehension, we may introduce predicate constants *U, V* together with the assertions

(1) $\forall x[U(x) \leftrightarrow (A \vee x = 0)]$ $\forall x[V(x) \leftrightarrow (A \vee x = 1)]$

winds up with a map $f \in J^I$ for which $k(i) \in A_{if(i)}$ for all $i \in I$. This is precisely what is demanded by (*).

In a non-extensional constructive framework such as CTT, the axiom of choice is compatible with intuitionistic logic, that is, with the non-affirmation of the law of excluded middle. But in 1975 Diaconescu showed that, in extensional frameworks such as topos theory or set theory, the usual formulations of the axiom of choice imply the law of excluded middle, so making logic classical. And Martin-Löf's analysis shows that, in CTT, the imposition of (a form of) extensionality on the axiom of choice will enable Diaconescu's theorem to become applicable, again yielding classical logic. (But if the axiom of choice is formulated within set theory or topos theory in the "harmless" (indeed mathematically useless) way (+), it is perfectly compatible with intutionistic logic.)

That extensionality in some form is required to derive Diaconescu's theorem can be observed in a number of different ways in addition to Martin-Löf's penetrating analysis. I offer two here.

1. Let $\mathscr{L}$ be a second-order language with individual variables $x, y, z, \ldots$, predicate variables $X, Y, Z, \ldots$ and second-order function variables $F, G, H, \ldots$. Here a second-order function variable F may be applied to a predicate variable X to yield an individual term FX. The scheme of sentences

Writing $f = \pi_2 \circ u(k)$, it follows from (#) that

$$f \in J^I \text{ and } k \in \prod_{i \in I} A_{if(i)},$$

whence

$$k \in \bigcup_{f \in J^I} \prod_{i \in I} A_{if(i)}.$$

So we have derived (*).

What is really going here appears to be the following. Under the epimorphism

$$\prod_{i \in I} \coprod_{j \in J} A_{ij} \twoheadrightarrow \prod_{i \in I} \bigcup_{j \in J} A_{ij}$$

information is "lost", to wit, the identity , for a given member g of the domain of the epi, and an arbitrary $i \in I$, of the $j \in J$ for which $g(i) \in A_{ij}$. The map u furnished by **EQ** essentially resupplies that information. So starting with $k \in \prod_{i \in I} \bigcup_{j \in J} A_{ij}$, if one applies u to it, and then applies to the result the isomorphism whose existence is ensured to exist by **IAC,** one

Each $k \in \prod_{i\in I}\bigcup_{j\in J} A_{ij}$ may be identified with the ≈-equivalence class $\{g\colon \pi_1\circ g = k\} = \tilde{k}$. Using **EQ**, choose a system of unique representatives from the ≈-equivalence classes. This amounts to introducing a map

$$u\colon \prod_{i\in I}\bigcup_{j\in J} A_{ij} \to \prod_{i\in I}\coprod_{j\in J} A_{ij}$$

for which $u(k) \in \tilde{k}$, i.e.

$$(**) \qquad \pi_1\circ u(k) = k,$$

for all $k \in \prod_{i\in I}\bigcup_{j\in J} A_{ij}$.

Now to establish (*), we take any $k \in \prod_{i\in I}\bigcup_{j\in J} A_{ij}$. Then under the natural isomorphism between $\prod_{i\in I}\coprod_{j\in J} A_{ij}$ and $\coprod_{f\in J^I}\prod_{i\in I} A_{if(i)}$ specified in the set-theoretic version of **IAC**, $u(k)$ is correlated with the pair of maps

$$(\pi_1\circ u(k)\ ,\ \pi_2\circ u(k)),$$

i.e., using (**), with

$$(k\ ,\ \pi_2\circ u(k)).$$

$$\prod_{i\in I}\bigcup_{j\in J}A_{ij} = \bigcup_{f\in J^I}\prod_{i\in I}A_{if(i)}\,.$$

which is in turn equivalent to

(*)
$$\prod_{i\in I}\bigcup_{j\in J}A_{ij} \subseteq \bigcup_{f\in J^I}\prod_{i\in I}A_{if(i)}\,.$$

I shall present a natural derivation within set theory of (*) from **IAC** and **EQ**, so providing what seems to me a purely set-theoretical formulation of Martin-Löf's argument.

First observe that there is a natural epimorphism

$$\prod_{i\in I}\coprod_{j\in J}A_{ij} \twoheadrightarrow \prod_{i\in I}\bigcup_{j\in J}A_{ij}$$

given by

$$g \mapsto \pi_1 \circ g$$

Write ≈ for the equivalence relation on $\prod_{i\in I}\coprod_{j\in J}A_{ij}$ given by

$$g \approx h \Leftrightarrow \pi_1 \circ g = \pi_1 \circ h.$$

coproduct (disjoint union) $\coprod$. (Here $\coprod_{i\in I} A_i$ may be identified with $\bigcup_{i\in I} A_i \times \{i\}$.)

Under this interpretation **IAC** may be taken to assert the existence, for any doubly indexed family of sets $\{A_{ij} : i \in I, j \in J\}$, of an isomorphism

(+) $$\prod_{i\in I} \coprod_{j\in J} A_{ij} \cong \coprod_{f\in J^I} \prod_{i\in I} A_{if(i)}.$$

The requisite, indeed canonical, isomorphism is easily supplied in the form of the map

$$g \mapsto (\pi_1 \circ g, \pi_2 \circ g) = g^*,$$

where π_1, π_2 are the projections of ordered pairs onto their first and second coordinates.
Note that

(#) for $g \in \prod_{i\in I} \coprod_{j\in J} A_{ij}$, g^* is a pair of functions (e, f) with $f \in J^I$ and $e \in \prod_{i\in I} A_{if(i)}$.

Now **CAC** can be shown, in standard (intuitionistic) set theory, to be equivalent to the assertion that, for any doubly indexed family of sets $\{A_{ij} : i \in I, j \in J\}$,

$$\forall x \in A \exists y \in B\, R(x,y) \to \exists f\colon A \to B\, [\mathrm{Ext}(f) \wedge \forall x \in A\, R(x, fx)].$$

The equivalence between **CAC** and **EAC,** be it noted, is established *within CTT* where (**1**), i.e., **IAC**, is already provable. There the equivalence is a nontrivial assertion. In set theory, on the other hand, not only are **CAC** and **EAC** equivalent, but they are themselves both equivalent to **IAC.** It becomes natural then to ask: how can Martin-Löf's argument be presented in *set theory* without courting triviality?

I believe this can be done by noting that Martin-Löf also establishes the equivalence, in CTT, of **CAC** with the assertion that unique representatives can be picked from the equivalence classes of any given equivalence relation. Let us abbreviate this as **EQ.** In deriving **CAC** (actually the equivalent **EAC,** but no matter) from **EQ,** Martin-Löf employs **IAC,** so establishing, in CTT, the implication

$$\mathbf{EQ} + \mathbf{IAC} \Rightarrow \mathbf{CAC}$$

The problem thus boils down to giving a faithful version of the argument for this implication within set theory.

To do this, **IAC** must be furnished with a *constructively valid set-theoretical* formulation. This can be achieved by invoking the "propositions as types" doctrine, according to which $\forall$ is interpreted as Cartesian product $\prod$ and $\exists$ as and made explicit the further assumptions needed to carry through his proof of the well-ordering theorem. These assumptions constituted the first explicit presentation of an axiom system for set theory.

It has been observed recently by Per Martin-Löf that Zermelo's 1904 formulation of the axiom of choice, i.e. (**1**), which Martin-Löf calls the *intensional* axiom of choice **IAC**, is actually quite unobjectionable from the more refined constructive point of view that has emerged since Zermelo's time. This is because, according to the interpretation of the quantifiers in (a major current version of) constructivism, constructive type theory CTT, the consequent of the implication of (**1**) means *nothing more than its antecedent.* Indeed, in CTT, the assertability of an alternation of quantifiers $\forall x \exists y R(x,y)$ means precisely that one is given a function f for which $R(x,fx)$ holds for all x. Martin-Löf goes on to show that this is decidedly *not* the case for Zermelo's abovementioned *second* formulation of the axiom of choice in 1908, the combinatorial axiom of choice.

Martin-Löf shows that, in CTT, **CAC** is equivalent to what he terms the *extensional* axiom of choice (**EAC**). This may be stated as follows. Let A and B be two sets carrying equivalence relations $=_A$ and $=_B$ respectively. A function f: $A \to B$ is called *extensional,* Ext(f),if $\forall x x' \in A \ (x =_X x' \to fx =_Y fx')$. Then **EAC** may be stated: for any relation R between A and B,

(1) $$\forall x\in A\exists y\in B\ R(x,y) \to \exists f\colon A \to B\ \forall x\in A\ R(x, fx)$$

and second, the assertion that for each collection $\mathscr{S}$ of nonempty sets, there is a function f—a *choice function* on $\mathscr{S}$— which associates with each member X of $\mathscr{S}$ a unique element $f(X)$ of X.

Zermelo's original purpose in introducing the axiom of choice was (as is well-known) to establish a central principle of Cantor's set theory, namely, that every set admits a well-ordering and so can also be assigned a cardinal number. His introduction of the axiom, as well as the use to which he put it, attracted considerable criticism from the mathematicians of the day. The chief objection raised was to what some saw as its highly non-constructive, even idealist, character: while the axiom asserts the possibility of making a number of—perhaps even an uncountable number—of arbitrary "choices", it gives no indication whatsoever of how these latter are actually to be effected, of how, otherwise put, choice functions can actually be *defined.* This was particularly objectionable to mathematicians of a "constructive" bent such as the so-called French Empiricists Baire, Borel and Lebesgue, for whom a mathematical object could be asserted to exist only if it can be uniquely defined. Zermelo's response to his critics came in the form in two papers in 1908 in which he reformulated the Axiom of Choice in terms of choice sets, as remarked above, element. The result is a choice set for $\mathscr{B}$. This argument, suitably refined, yields a precise derivation of the axiom of choice from the set-theoretical principle known as Zorn's lemma.

Let us call the Zermelo's 1908 formulation the *combinatorial* axiom of choice **CAC**.

Zermelo had in fact offered an earlier formulation (in 1904) of the axiom of choice in terms of what he called *coverings*, the contemporary term for which is *choice function.* Here he starts with an arbitrary set *M* and uses the symbol *M′* to denote an arbitrary nonempty subset of *M*, the collection of which he denotes by **M.** He continues:

Imagine that with every subset M′ there is associated an arbitrary element m_1' *, that occurs in M′ itself; let* m_1' *be called the "distinguished" element of M′. This yields a "covering"* γ *of the set* **M** *by certain elements of the set M. The number of these coverings is equal to the product* [of the cardinalities of all the subsets *M′*] *and is certainly different from 0.*

The last sentence of this quotation—which asserts, in essence, that coverings always exist for the collection of nonempty subsets of any (nonempty) set—is Zermelo's first formulation of the axiom of choice. This may be equivalently reformulated in two ways: first, the implication, for any relation *R* between sets *A, B*

The Axiom of Choice in the Foundations of Mathematics

John L. Bell

The principle of set theory known as the *Axiom of Choice* (**AC**) has been hailed as "probably the most interesting and, in spite of its late appearance, the most discussed axiom of mathematics, second only to Euclid's axiom of parallels which was introduced more than two thousand years ago." [1] From the fulsomeness of this description one might expect the axiom to constitute as startling an assertion as, say, the Principle of the Constancy of the Velocity of Light or the Heisenberg Uncertainty Principle. But in fact the Axiom of Choice in its usual formulation seems humdrum, even self-evident. As stated by Zermelo in 1908 it amounts to the claim that, given any collection of mutually disjoint nonempty sets, it is possible to assemble a new set—a *choice set*—containing exactly one member from each.

In this formulation the Axiom of Choice can be furnished with a "combinatorial" justification along the following lines. Given a family $\mathscr{B}$ of mutually disjoint nonempty sets, call a subset $S \subseteq \cup\mathscr{B}$ a *selector* for $\mathscr{B}$ if $S \cap B \neq \varnothing$ for all $B \in \mathscr{B}$. Clearly selectors for $\mathscr{B}$ exist; $\cup\mathscr{B}$ itself is an example. Now one can imagine taking a selector S for $\mathscr{B}$ and "thinning out" each intersection $S \cap B$ for $B \in \mathscr{B}$ until it contains just a single

[1] Fraenkel, Bar-Hillel and Levy